Evolution of the energy spectrum among a large number of internal waves in the deep ocean

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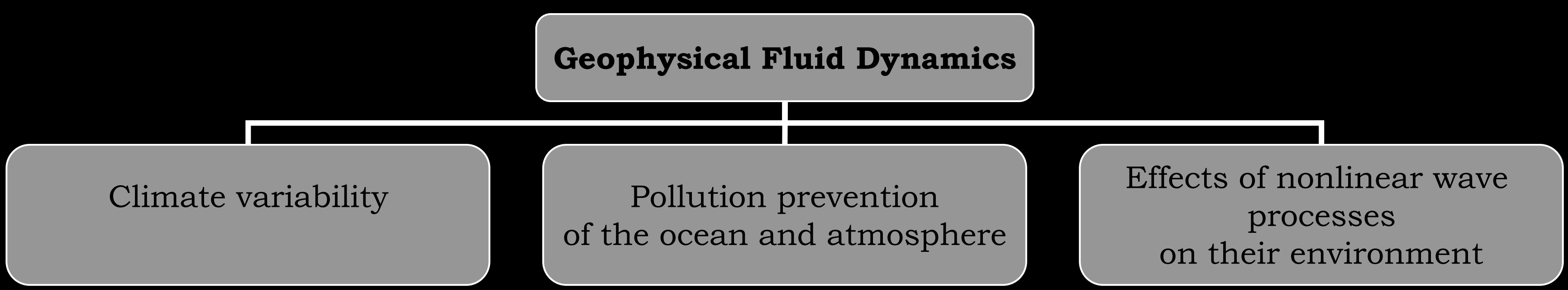
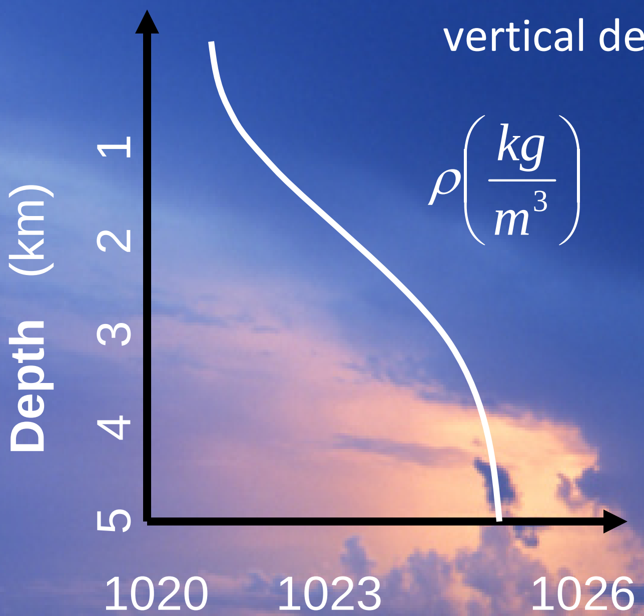
Present research interests and experience

- Existence and uniqueness of solutions for nonlinear PDEs
- Numerical solutions of Boundary Value Problems
- Mathematical modeling
- Solitary waves in inhomogeneous medium
- Stability analysis of nonlinear PDEs (NS)
- Lie group analysis of ODEs and PDEs
- Geophysical Fluid Dynamics



Geophysical Fluid Dynamics

- **The object:** study of naturally occurring large-scale flows on the Earth
- **Two features:** rotation of the Earth;
  vertical density stratification of the medium



Density at surface between $1000-1030\ \frac{kg}{m^3}$

At great depths (5 km) can have $1050\ \frac{kg}{m^3}$

Set

$$\boxed{\rho = \rho_0 + \rho'}$$

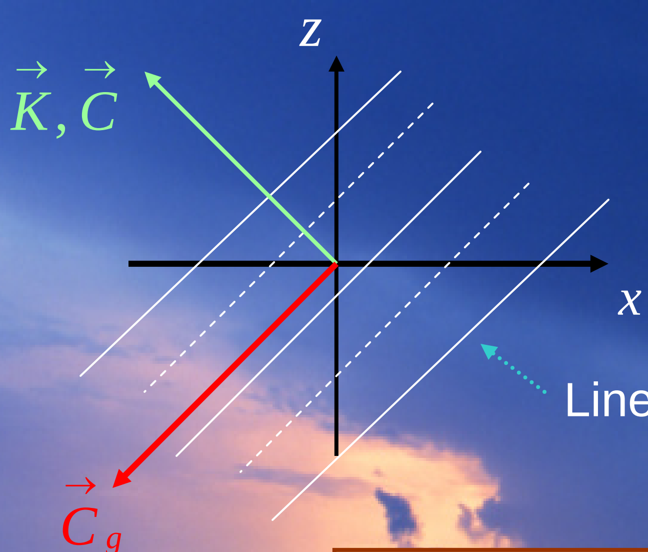
$$\rho' << \rho_0 \quad \Rightarrow \quad \rho\frac{D\vec{u}}{Dt} \approx \rho_0\frac{D\vec{u}}{Dt}$$

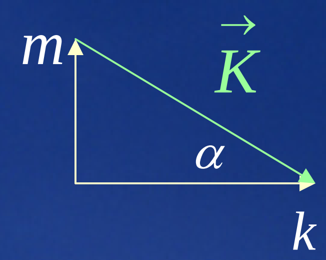
Boussinesq Approximation

Internal Gravity Waves

- Waves that exist in stratified fluids due to gravitational restoring forces

Wave propagating in 2D: $w = w_0 e^{i(kx + mz - \omega t)}$





$$\cos\alpha = \frac{k}{\vec{K}}$$

PDE for w $\Longrightarrow$ $\omega^2 = \dfrac{N^2k^2 + f^2m^2}{k^2 + m^2} = N^2\cos^2\alpha + f^2\sin^2\alpha$

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$$

ω function of direction $\vec{K}$ and not of the magnitude $\vec{K}$

$\Longleftrightarrow$ sharp contrast

ω function of magnitude only for surface waves

$$|f| < |\omega| < |N|$$

Geophysical Fluid Dynamics (GFD)

Internal waves (IWs) in GFD

Generation

Propagation

Nonlinear interactions

Dissipation

Energy transfer in internal wave field

rate of ocean mixing

GM spectrum

- Plays an important role in climate variability

-- Most uncertain component of ocean General Circulation Model

Describes internal wave field that is horizontally isotropic and vertically symmetric

Remarkable fact: GM has much the same shape in the ocean

Nonlinear wave interactions smooth out any spectral irregularity by redistributing energy within the spectrum

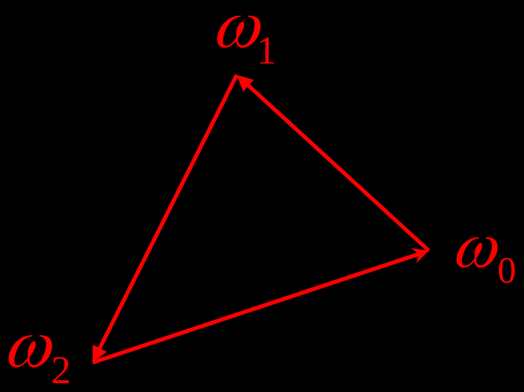
Resonant triads

- A resonant wave is a wave satisfying the dispersion relation and propagating at the frequency and the wave-number produced by two primary waves

- A triad of internal waves can interact resonantly among themselves if the linear dispersion relation satisfies

$$\overrightarrow{K}_1 \pm \overrightarrow{K}_2 \pm \overrightarrow{K}_0 = 0,$$

$$\omega_1 \pm \omega_2 \pm \omega_0 = 0$$



Previous studies

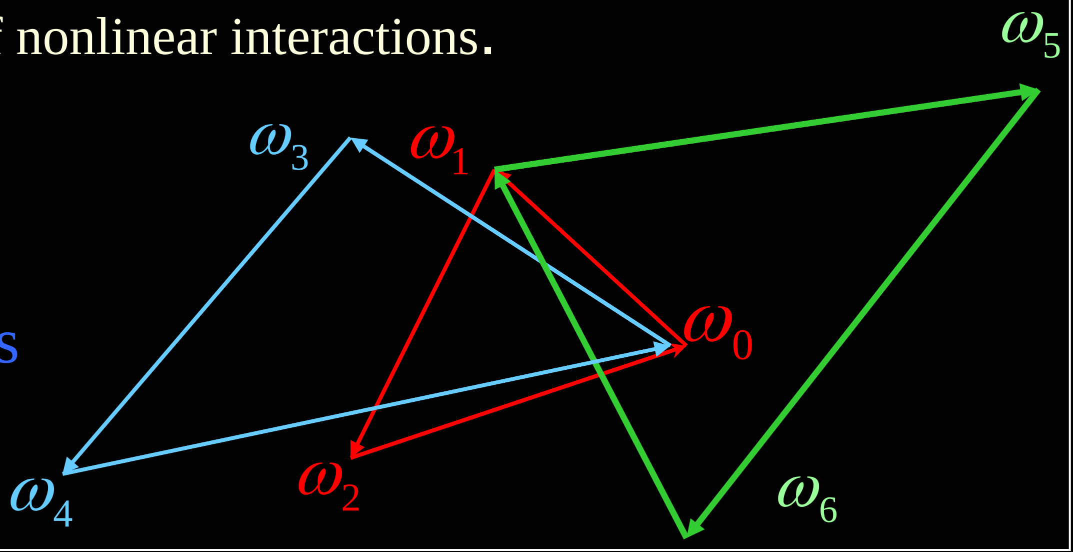
- Analytic approach:

  The energy distribution is well understood for the case of a single triad

- Statistical approach and Numerical analysis

  Do not explain the nature of nonlinear interactions.

Need a better model
based on physics of IWs



Resonant interactions

$$\nabla^2\psi_t - g\rho_x - fv_z = J(\psi,\nabla^2\psi),$$
$$v_t + f\psi_z = J(\psi,v),$$
$$\rho_t + \frac{N^2}{g}\psi_x = J(\psi,\rho).$$

$$u = \psi_z, \quad w = -\psi_x$$
$$J(a,b) = a_x b_z - a_z b_x$$

$$\rho = \overline{\rho}(z) + \rho' \qquad N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}$$

background density

Buoyancy frequency

The advective term in the density equation is written in the linearized form:

$$w\frac{d\overline{\rho}}{dz} = -\rho_0 N^2 \frac{w'}{g}$$

$\Rightarrow$ the rate of change of the density in a point is only due to the vertical advection of the undisturbed density distribution $\overline{\rho}(z)$

Look for solutions $$(\psi, v, \rho) = \sum_i \varepsilon^i (\psi_i, v_i, \rho_i)$$

$$\psi_1 = \sum_i A_i \frac{\omega_i}{k_i} \sin\theta_i \sin(m_i z),$$

$$\rho_1 = \frac{N^2}{g} \sum_i A_i \sin\theta_i \sin(m_i z)$$

$$v_1 = -f \sum_i A_i \frac{m_i}{k_i} \cos\theta_i \cos(m_i z)$$

$$\theta_i = k_i x - \omega_i t + \varphi_i$$

Dispersion relation

$$\boxed{\omega_i^2 = \frac{N^2 k_i^2 + f^2 m_i^2}{k_i^2 + m_i^2}}$$

The nonlinear interactions become evident only in the second order problem:

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_2 + N^2\frac{\partial^2\psi_2}{\partial x^2} + f^2\frac{\partial^2\psi_2}{\partial z^2} = F_2(\psi_1, v_1, \rho_1)$$

Multiple time scale analysis: $A_i = A_i(\tau)$, $\tau = \varepsilon t$ - slow time scale

$$F_2 = \frac{1}{4}\sum_i\sum_j A_i A_j\left[\sum_{l=1}^{4} S_l\right] - S_r(A_i, \omega_i, k_i, m_i, \tau)$$

$$S_l = \pm\left[\frac{\omega_i\omega_j}{k_i k_j}\left|\vec{k}_j\right|^2(\omega_i \pm \omega_j) + N^2\frac{\omega_i}{k_i}(k_i \pm k_j) + f^2(m_i \pm m_j)\frac{\omega_i m_j}{k_i k_j}\right]\times$$
$$(m_i k_j \pm m_j k_i)\cos(\theta_i \pm \theta_j)\sin([m_i \pm m_j]z)$$

$$S_r = \sum_i\left[\frac{\omega_i^2}{k_i}\left|\vec{k}_j\right|^2 + N^2 k_i + f^2\frac{m_i^2}{k_i}\right]\left(\frac{dA_i}{d\tau}\right)\cos\theta_i\sin(m_i z)$$

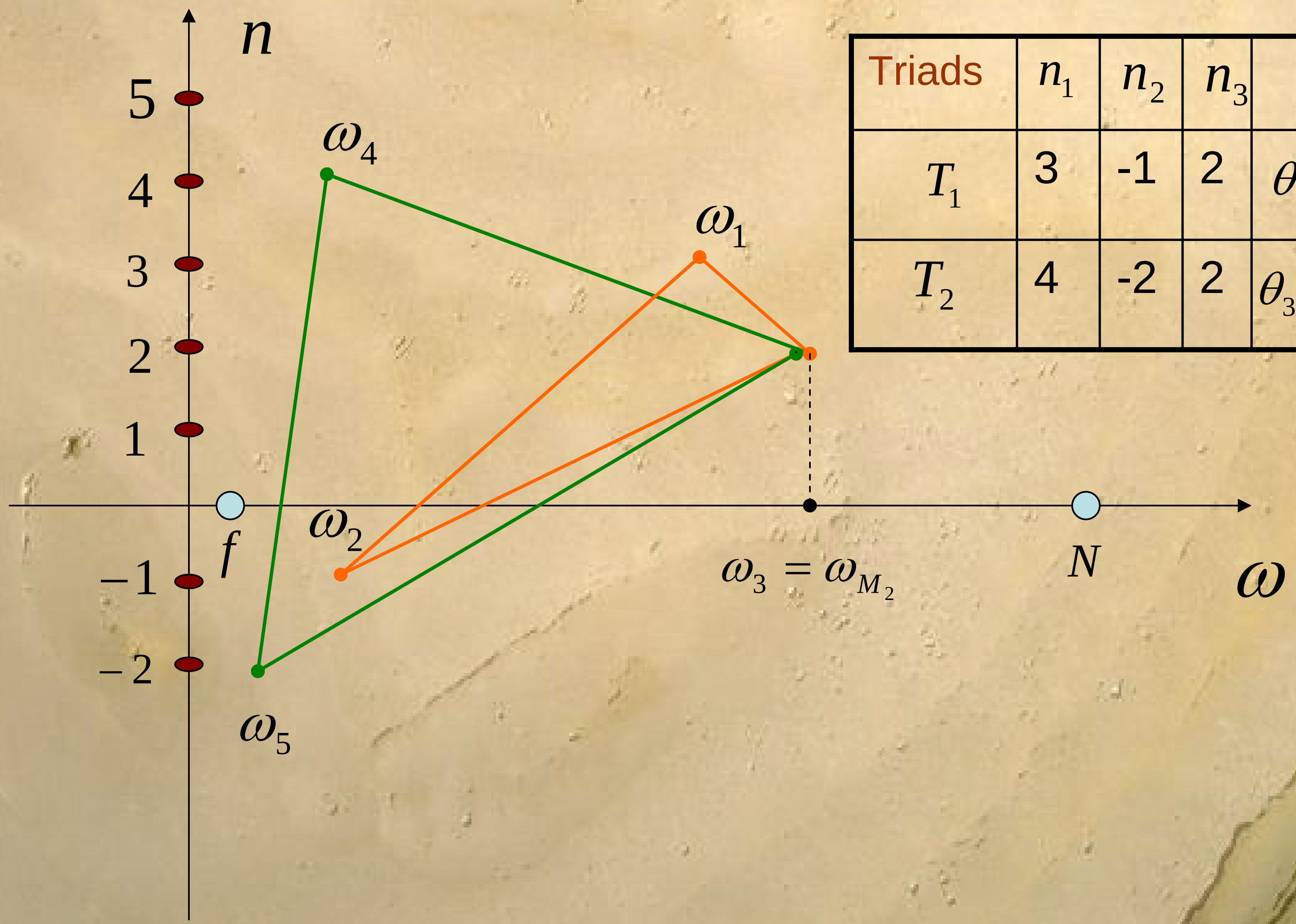
For resonant terms

$$2\sum_i \frac{\omega_i^2}{k_i}\left|\overrightarrow{k_j}\right|^2 A_i^{/} \cos\theta_i \sin(m_i z) =$$

$$= -\frac{1}{4}\sum_i\sum_j A_i A_j \left[\frac{\omega_i\omega_j}{k_i k_j}\left|\overrightarrow{k_j}\right|^2(\omega_i+\omega_j) + N^2\frac{\omega_i}{k_i}(k_i+k_j) + f^2(m_i+m_j)\frac{\omega_i m_j}{k_i k_j}\right]\times$$

$$(m_i k_j - m_j k_i)\cos(\theta_i+\theta_j)\sin([m_i+m_j]z)$$

Example: Two resonant triads



Triads	n_1	n_2	n_3	
T_1	3	-1	2	$\theta_3 = \theta_1 + \theta_2$
T_2	4	-2	2	$\theta_3 = \theta_4 + \theta_5$

For this case,

$$\sum_{i=1}^{5}\frac{\omega_i^2}{k_i^2}\left|\overrightarrow{k_j}\right|^2 A_i^{/}\cos\theta_i\sin(m_i z)=$$

$$A_1A_2(B_{12}+B_{21})\cos\theta_3\sin(m_3 z)+A_4A_5(B_{45}+B_{54})\cos\theta_3\sin(m_3 z)+$$
$$A_1A_3(B_{13}+B_{31})\cos\theta_2\sin(m_2 z)+A_2A_3(B_{23}+B_{32})\cos\theta_1\sin(m_1 z)+$$
$$A_3A_4(B_{34}+B_{43})\cos\theta_5\sin(m_5 z)+A_3A_5(B_{35}+B_{53})\cos\theta_4\sin(m_4 z)$$

$$\boxed{B_{ij}=\frac{1}{2}\left[\frac{\omega_i\omega_j}{k_ik_j}\left|\overrightarrow{k_j}\right|^2(\omega_i+\omega_j)+N^2\frac{\omega_i}{k_j}(k_i+k_j)+f^2(m_i+m_j)\frac{\omega_i m_j}{k_ik_j}\right]\times(m_ik_j-m_jk_i)}$$

Evolution equations

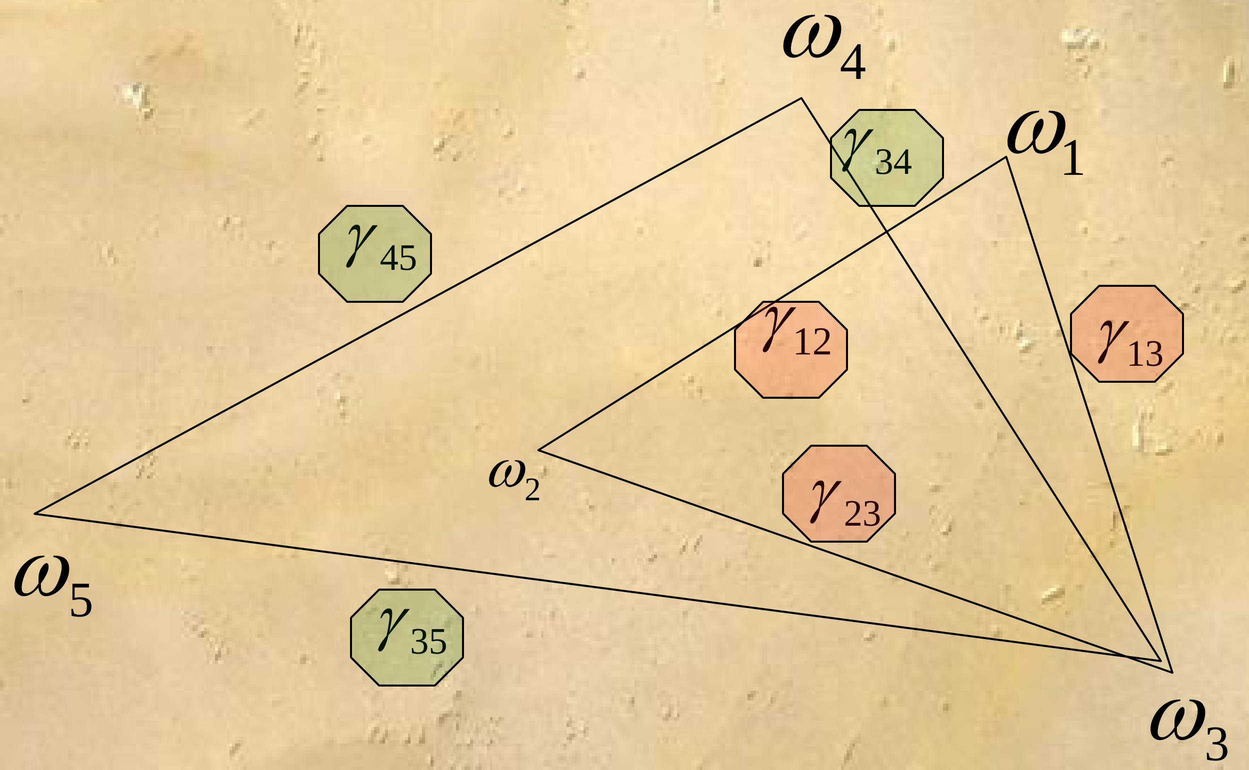
$$\frac{dA_1}{d\tau} = \gamma_{23} A_2 A_3$$

$$\frac{dA_2}{d\tau} = \gamma_{13} A_1 A_3$$

$$\frac{dA_3}{d\tau} = \gamma_{12} A_1 A_2 + \gamma_{45} A_4 A_5$$

$$\frac{dA_4}{d\tau} = \gamma_{35} A_3 A_5$$

$$\frac{dA_5}{d\tau} = \gamma_{34} A_3 A_4$$



$$\gamma_{ij} = \frac{\left(B_{ij} + B_{ji}\right) k_l}{\left|\overrightarrow{k_l}\right|^2 \omega_l^2}$$

Two questions

- Are resonant interactions in fact possible?
- How do the wave amplitudes evolve in time?

Solutions to the resonance conditions

Given

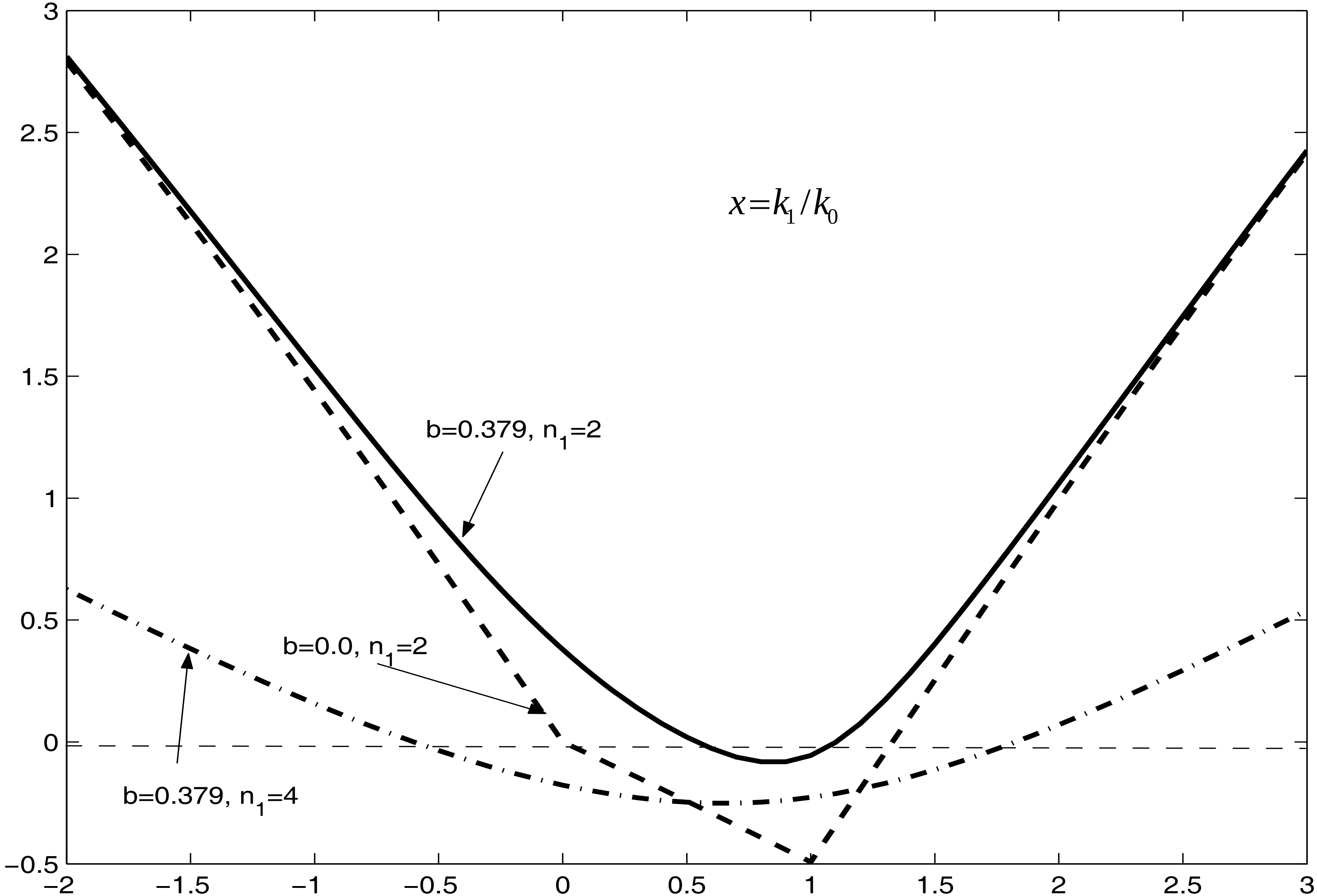
<u>Sum reactions</u>: $$\vec{K}_1 + \vec{K}_2 = \vec{K}_0, \quad \omega_1 + \omega_2 = \omega_0$$

The high frequency wave $\left(\omega_0, \vec{K}_0\right)$ is assumed to be known.

Synchronism condition: $$h_s(x) \equiv \sqrt{\frac{\left(\frac{x}{y}\right)^2 + b^2}{\alpha^2\left(\frac{x}{y}\right)^2 + 1}} + \sqrt{\frac{\left(\frac{1-x}{1-y}\right)^2 + b^2}{\alpha^2\left(\frac{1-x}{1-y}\right)^2 + 1}} - \sqrt{\frac{1+b^2}{\alpha^2+1}} = 0$$

$$x = \frac{k_1}{k_0}, \quad y = \frac{m_1}{m_0}, \quad b^2 = \frac{f^2}{N^2\alpha^2} \qquad \alpha = \frac{k_0}{m_0}$$ - aspect ratio

The idea is to treat y as known and see if we can solve for x.



Resonant curves (resonance occurs at roots of $h_s(x)$) for sum reactions, $\omega_1 + \omega_2 = \omega_0$ for $x = \frac{k_1}{k_0}$ at the given values of $n_0 = 1, n_1 = 2, n_1 = 4$ and $\omega_0 = \omega_{M_2} = 1.4 \times 10^{-4} s^{-1}$.

Solutions to the resonance conditions

Difference reactions: In this case wave 1 (one of the low frequency waves) is known, and we choose the mode number of wave 0, i.e., after switching the indices on waves 0 and 1, we have

$$\omega_1 - \omega_0 = \omega_2 \quad or \quad \omega_0 + \omega_2 = \omega_1$$

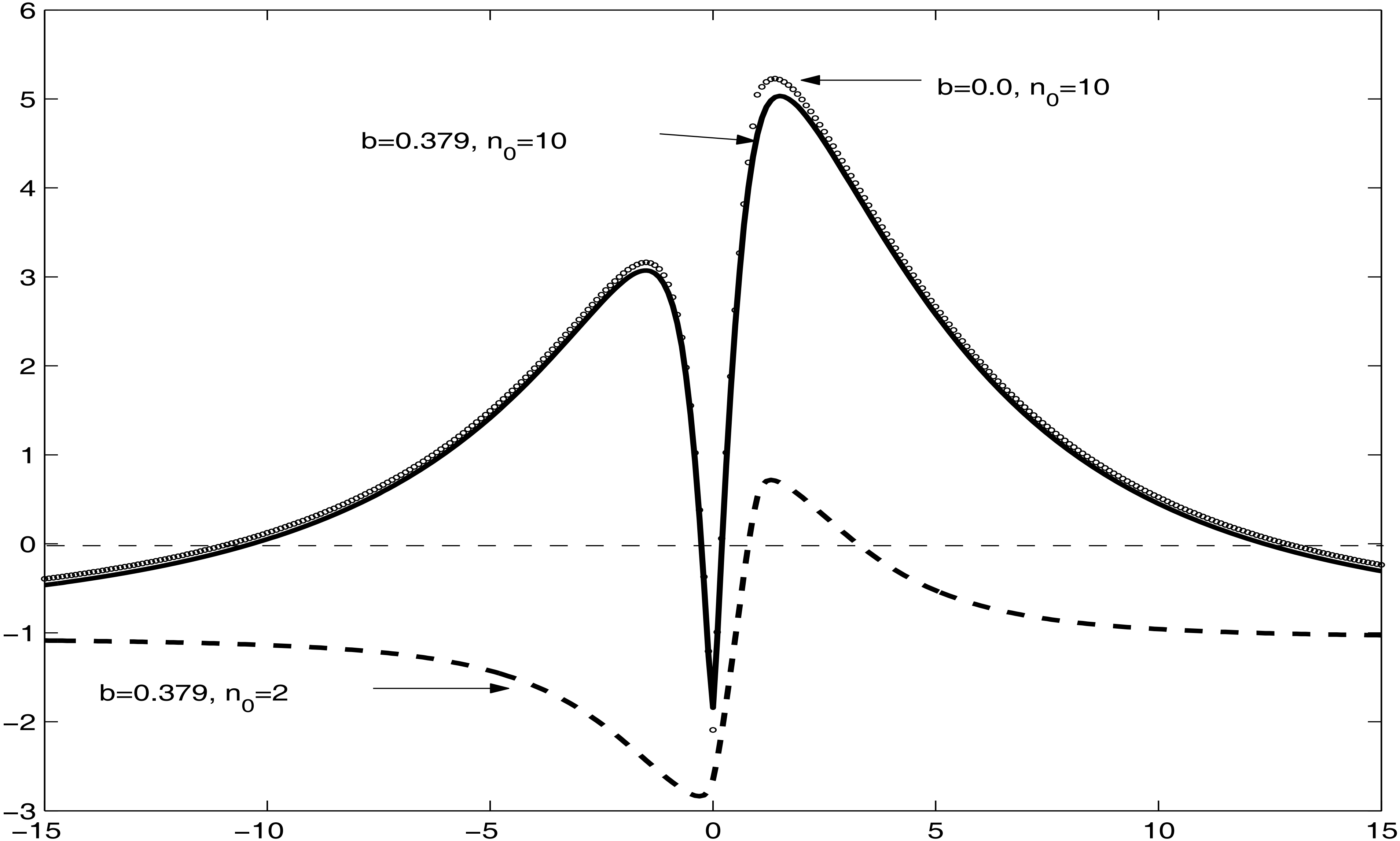
Given

Synchronism condition:

$$h_d(x) \equiv \sqrt{\frac{\left(\frac{x}{y}\right)^2 + b^2}{\alpha^2\left(\frac{x}{y}\right)^2 + 1}} - \sqrt{\frac{\left(\frac{x-1}{y-1}\right)^2 + b^2}{\alpha^2\left(\frac{x-1}{y-1}\right)^2 + 1}} - \sqrt{\frac{1+b^2}{\alpha^2+1}} = 0$$

[1] There may be as many as four solutions;

[2] While we could assume $n_1 > 0$ for the sum reactions, this is not true for difference reactions $\Rightarrow$ positive and negative values of n_1 must be considered.



Resonant curves (resonance occurs at roots of $h_d(x)$) for difference reactions,

$\omega_0 - \omega_1 = \omega_2$ for $x = k_1 / k_0$ at the given values of $n_0 = 2, n_0 = 10, n_1 = 1$

Resonant triad model

- Generates an arbitrary large number of resonant triads of IWs in (ω, n) space with frequencies spanning the range of possible frequencies;
- Generates the evolution equations for the amplitudes for each obtained resonant wave;
- Wave amplitudes are initialized on a chose of the initial energy spectrum;
- The obtained evolution equations to determine the temporal evolution of each wave are solved;
- The temporal evolution of energy distribution among the various possible wave numbers and frequencies based on the previous results is computed.

NOTE: The model can be used to determine the temporal evolution of an arbitrarily chosen initial energy spectrum.

Initial amplitudes expressed in terms of the GM spectrum:

$$\boxed{A_i(0) = \left[\frac{4E_i(0)}{\left(N^2 + f^2 \frac{m_i^2}{k_i^2}\right)} \right]^{\frac{1}{2}}} \qquad E_i(0) = E_{GM}\left(n; f + j\Delta\omega, f + (j-1)\Delta\omega\right)/ r_n$$

r_n - number of waves in each frequency band at given mode number.

$$\boxed{E_{GM} = H^2 N_0 N E_0 B(\omega) H(n)}$$

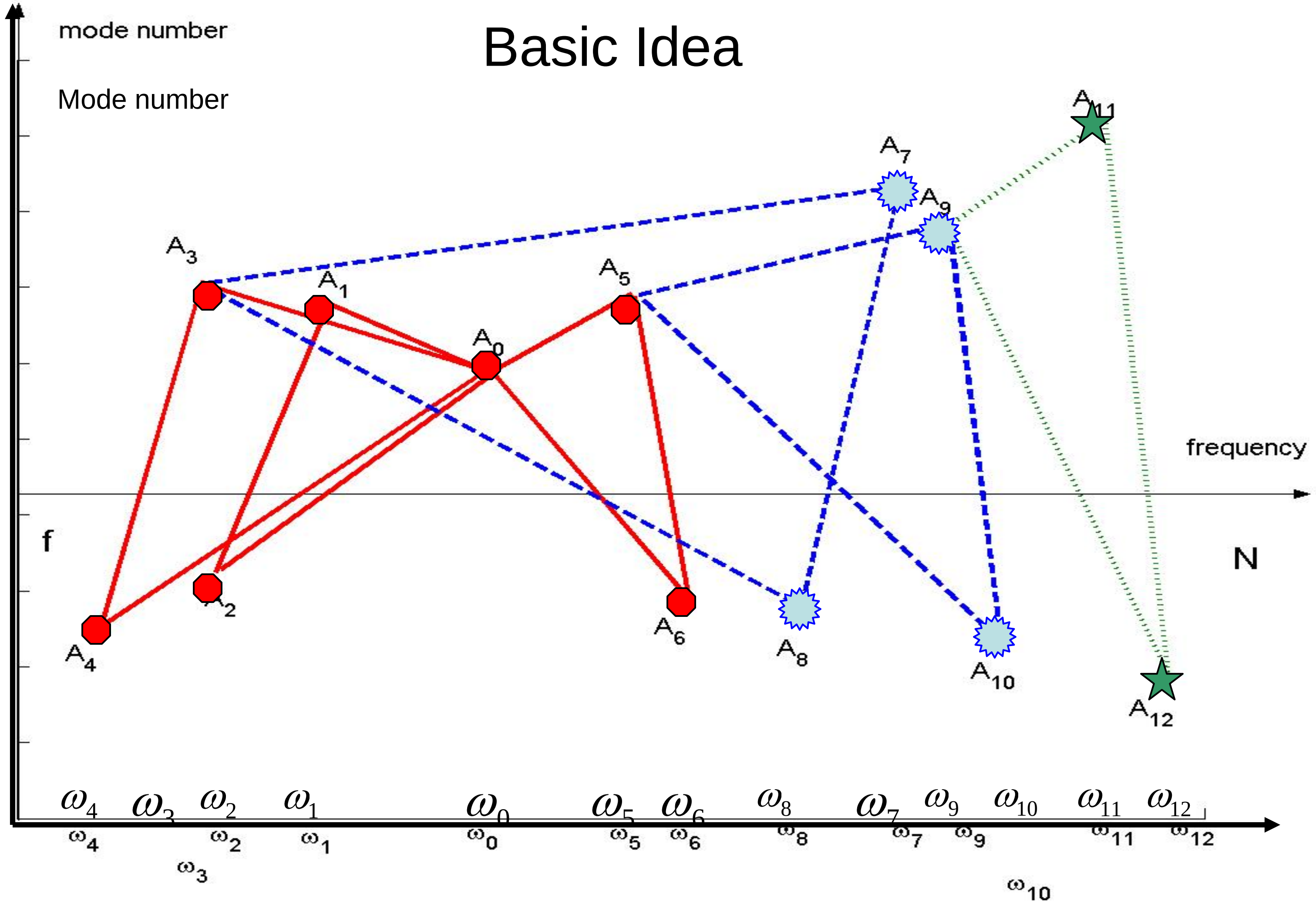
$$N(z) = N_0 e^{z/H}, \quad N_0 = 5.2 \times 10^{-3} s^{-1}, \quad H = 1.3km, \quad E_0 = 6 \times 10^{-5}.$$

Represent the initial energy distribution of a given mode number n in each frequency band by

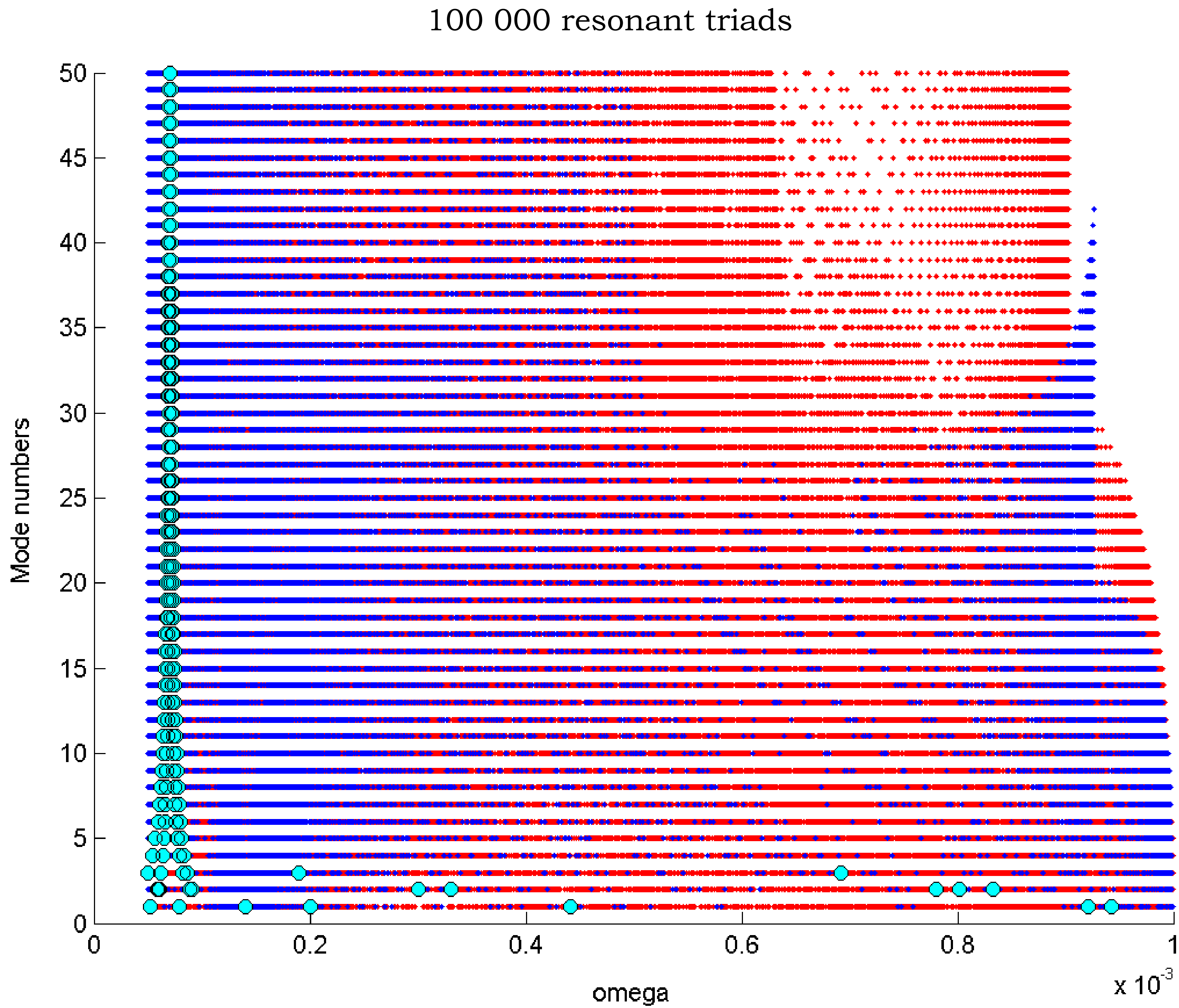
$$\boxed{\int_{f+(j-1)\Delta\omega}^{f+j\Delta\omega} E_{GM}(\omega; n) d\omega,} \qquad j = 1,\ldots, s$$

number of frequency bands

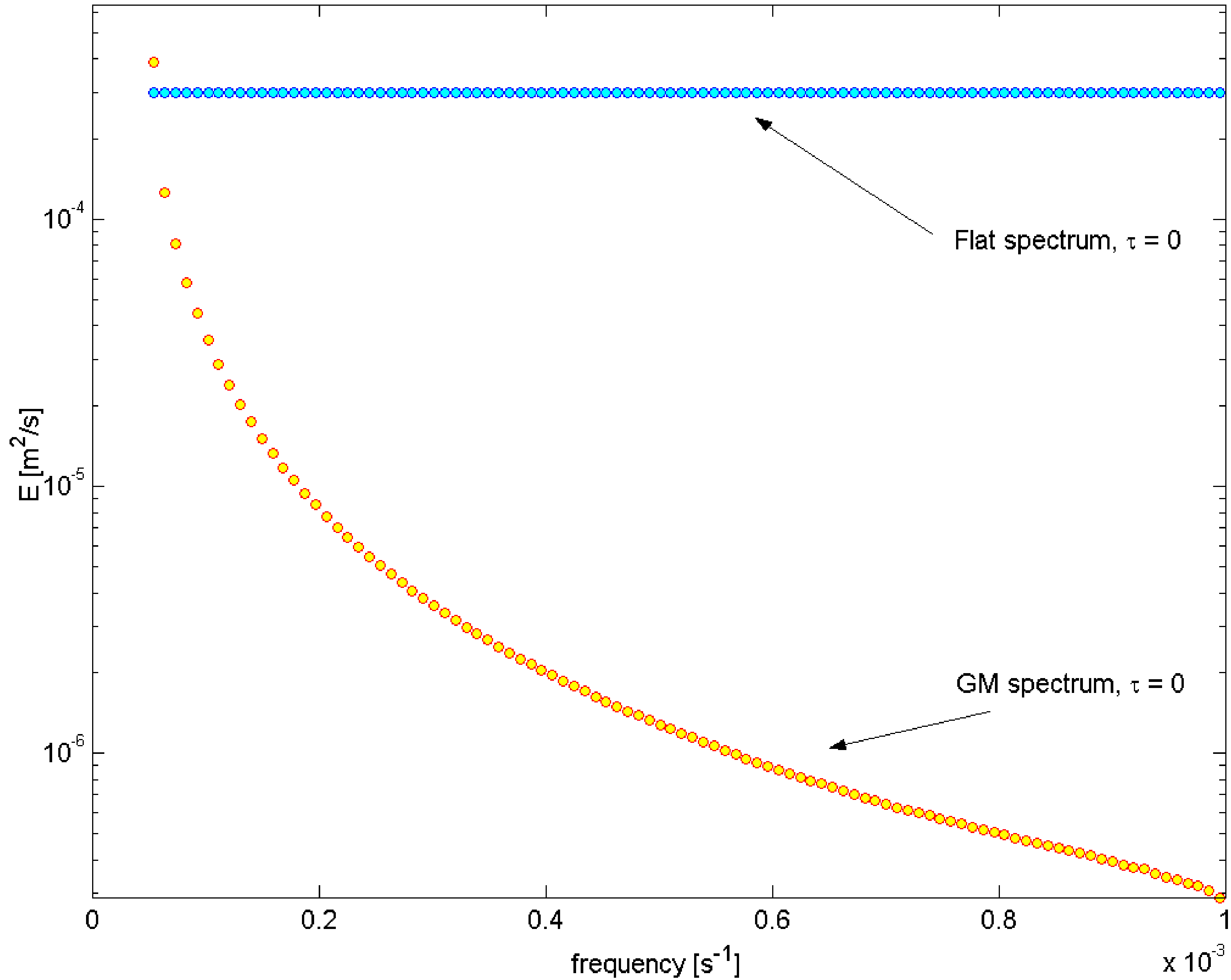
Basic Idea

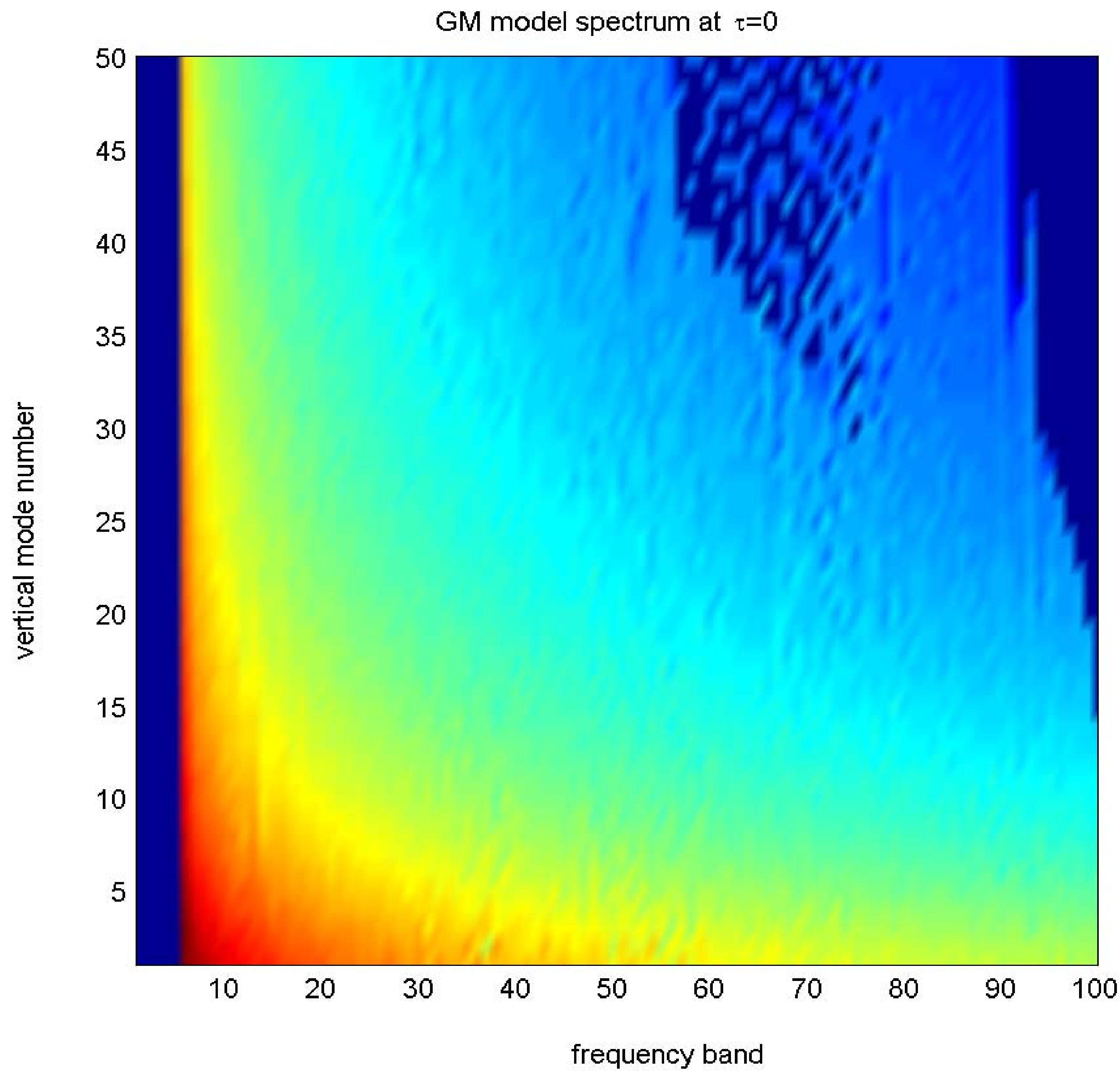


Schematic to illustrate the process of generation of resonant triads in $(\omega, n)-$ space. All triads are generated from initial base wave of frequency ω_0. The level 0 triads are plotted by a solid red line. Level 1 triads are plotted by a blue line and Level 2 triads are plotted by green dots.

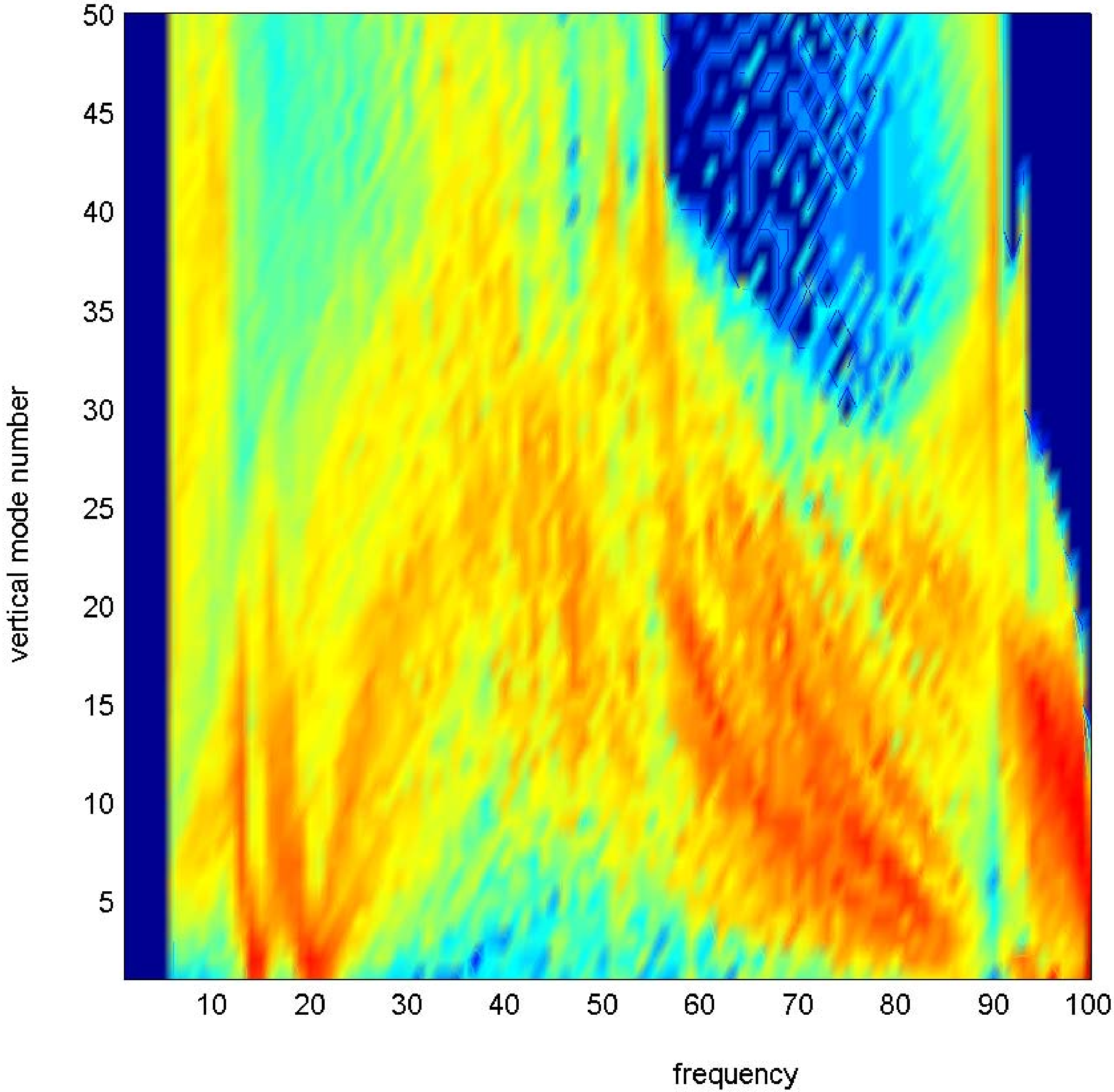

100 000 resonant triads
Mode numbers
omega
x 10^{-3}
0
5
10
15
20
25
30
35
40
45
50
0
0.2
0.4
0.6
0.8
1

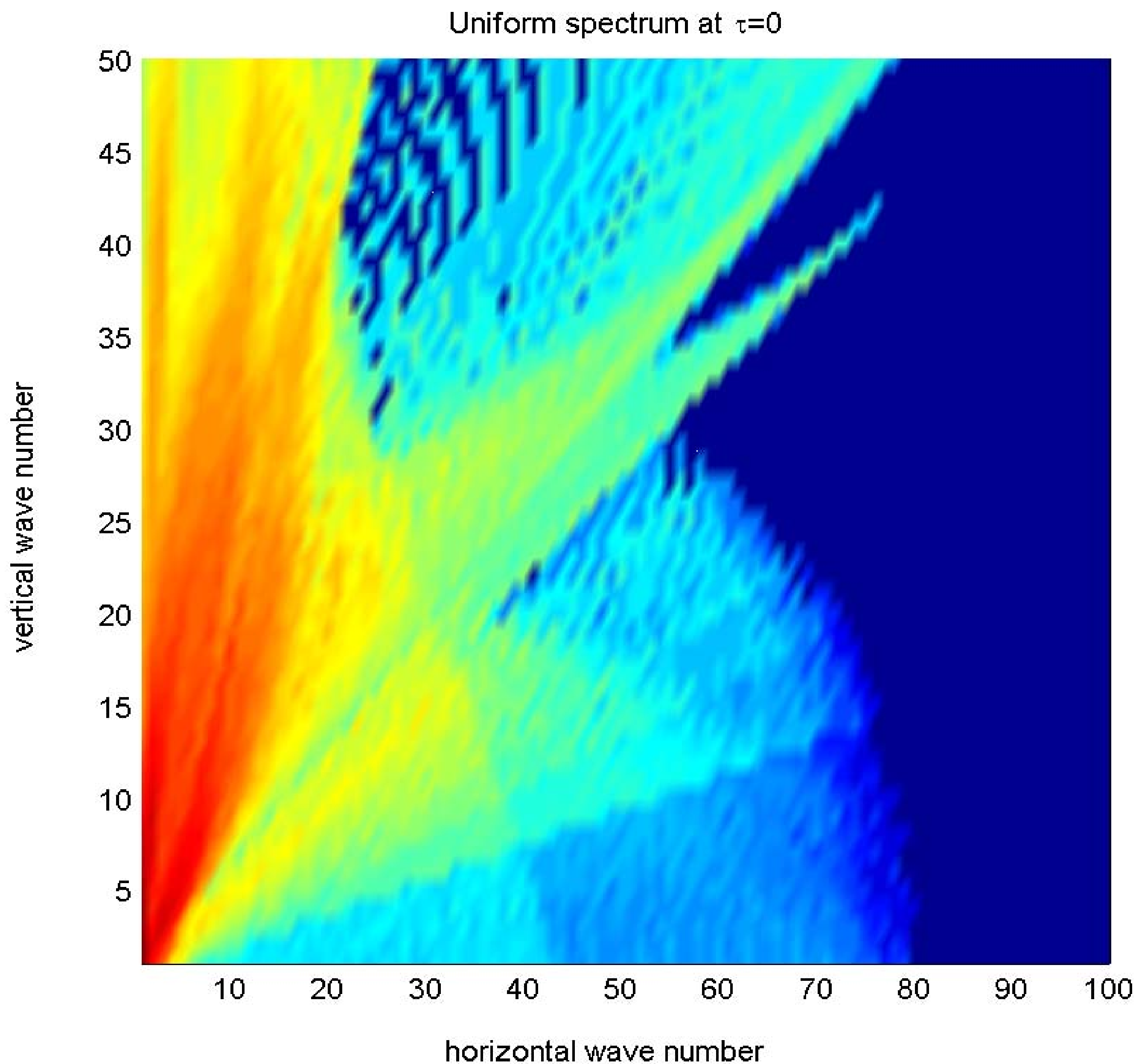
Two energy spectrums




GM model spectrum at τ=0
vertical mode number
frequency band
50
45
40
35
30
25
20
15
10
5
10
20
30
40
50
60
70
80
90
100

Uniform spectrum at τ=0

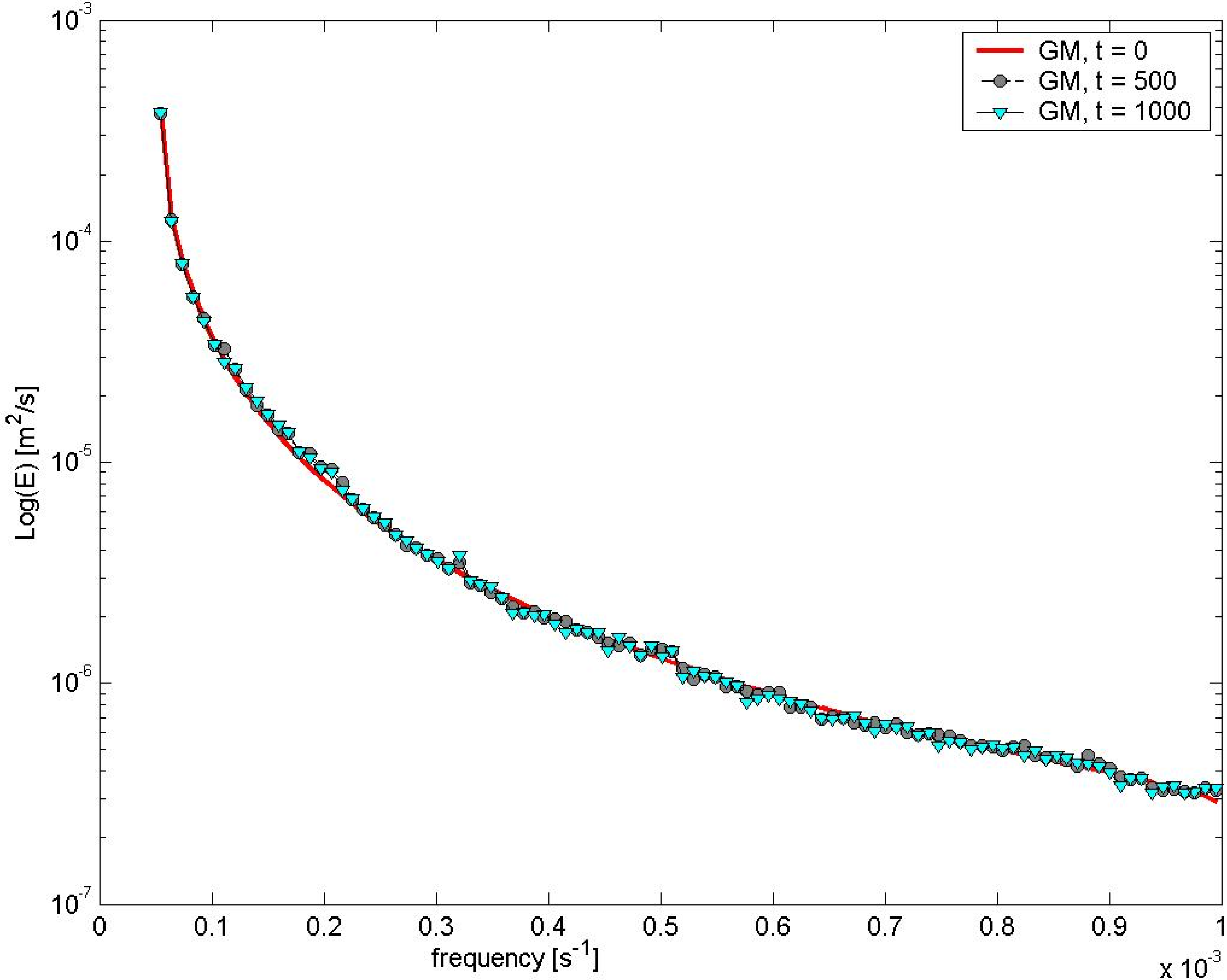




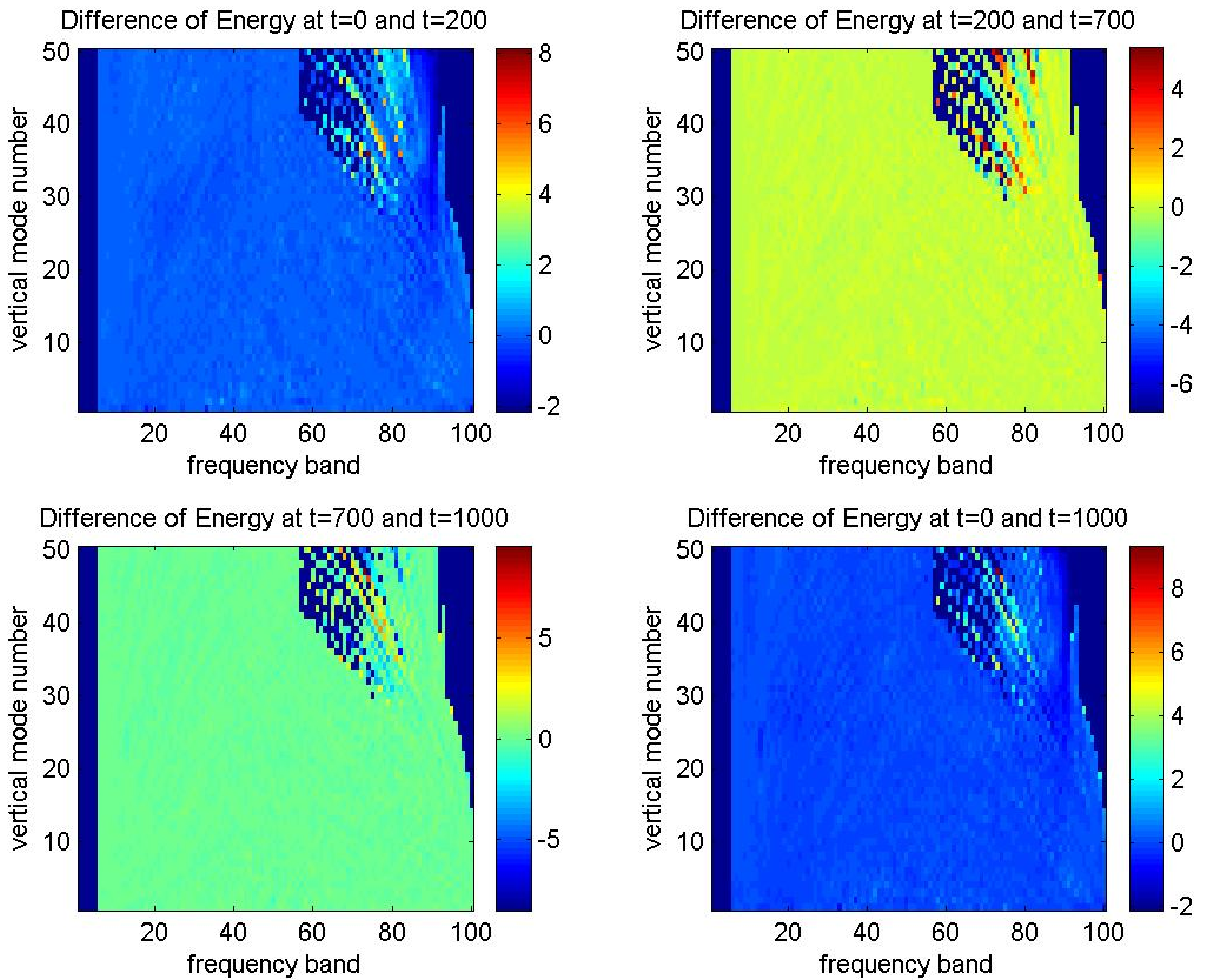
Uniform spectrum at τ=0
vertical wave number
horizontal wave number
50
45
40
35
30
25
20
15
10
5
10
20
30
40
50
60
70
80
90
100

Results

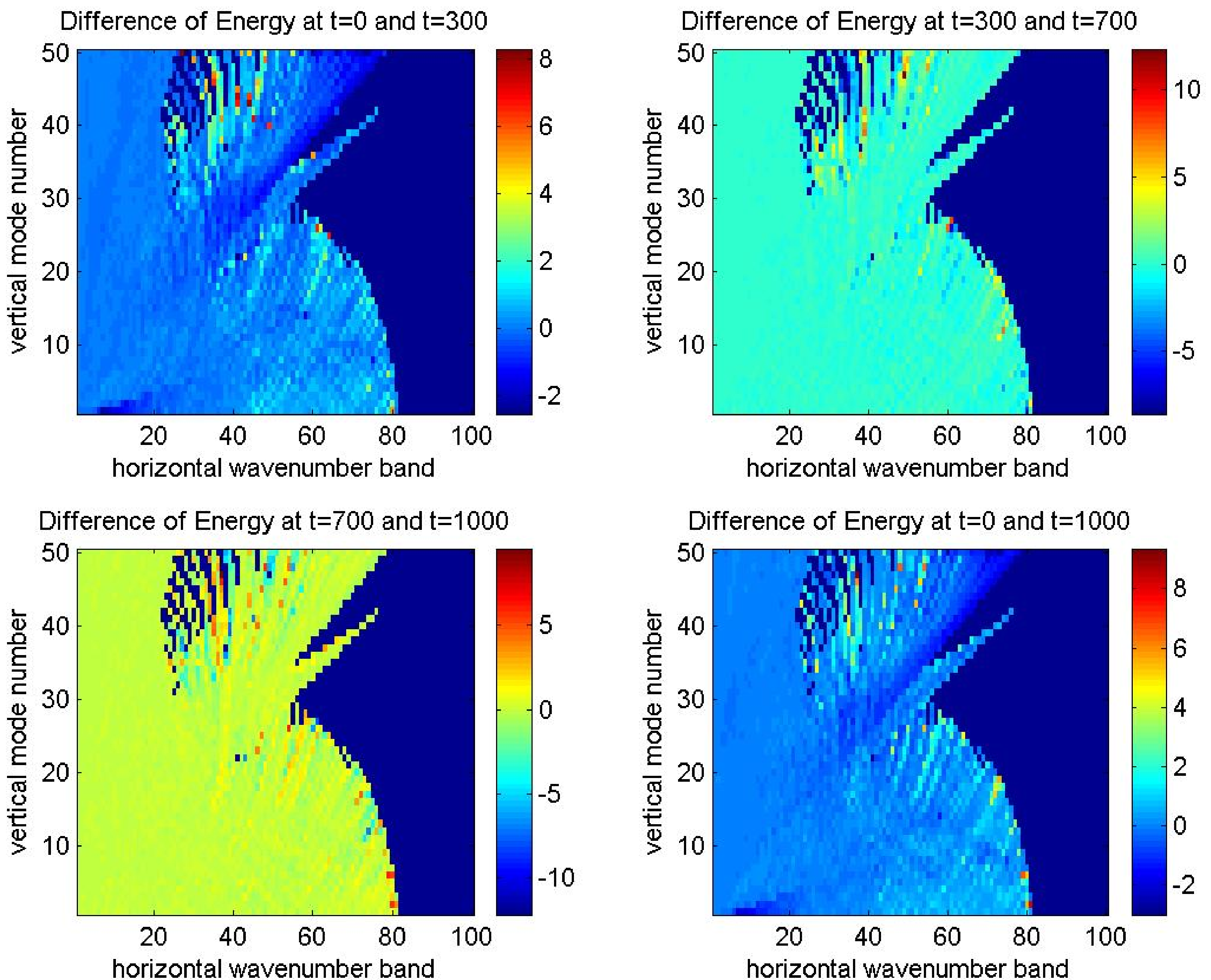
- Evolution of the wave’s amplitudes to predict the evolution of the internal wave energy spectrum
- Solve the system of 200 000 differential equations

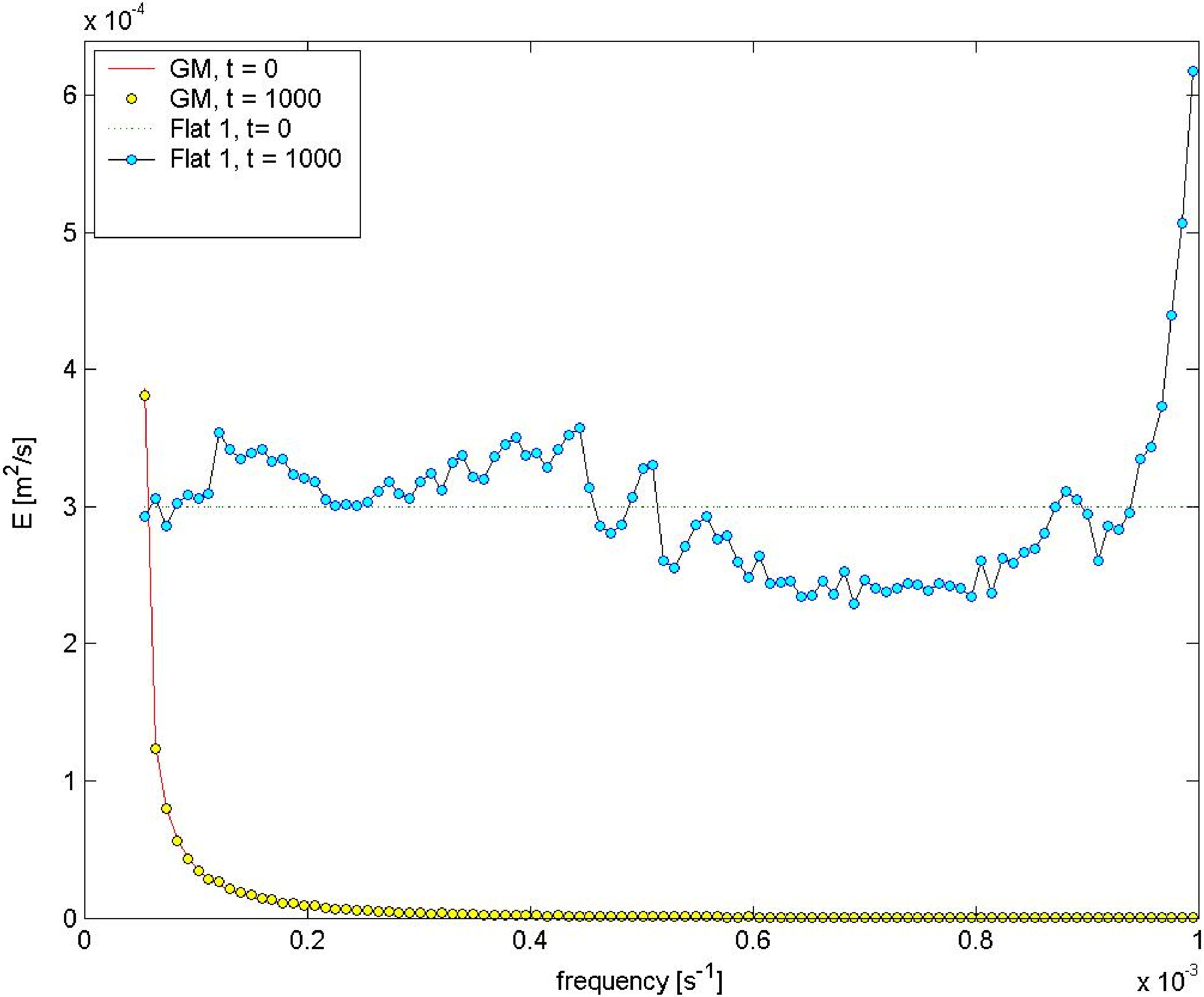


GM, t = 0
GM, t = 500
GM, t = 1000
Log(E) [m^2/s]
frequency [s^-1]
x 10^-3
10^-3
10^-4
10^-5
10^-6
10^-7
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Difference of Energy at t=0 and t=200
Difference of Energy at t=200 and t=700
Difference of Energy at t=700 and t=1000
Difference of Energy at t=0 and t=1000
vertical mode number
frequency band


Difference of Energy at t=0 and t=300
Difference of Energy at t=300 and t=700
Difference of Energy at t=700 and t=1000
Difference of Energy at t=0 and t=1000
vertical mode number
horizontal wavenumber band


x 10^-4
GM, t = 0
GM, t = 1000
Flat 1, t= 0
Flat 1, t = 1000
6
5
4
3
2
1
0
E [m^2/s]
0
0.2
0.4
0.6
0.8
1
frequency [s^-1]
x 10^-3

Other results & Conclusions

- Evolution of the GM spectrum at different latitudes;
- Forced waves (energy inflow);
- Modified GM spectrum;

GATE (Global Atlantic Tropical Experiment, 1974) shows a prominent tidal peak at the tidal frequency

- Oscillatory nature of the model;
- The GM model represents the stable state